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LETTER TO THE EDITOR

BOOTON'S PROBLEM RE-EXAMINED*

I. ELISHAKOFF

Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431-0981, *U.S.A.*

AND

P. COLAJANNI

Dipartimento di Ingegneria Strutturale e Geotecnica, Universita' di Palermo, Viale delle Scienze, I-90128, *Italy*

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1. INTRODUCTION

Random vibration theory of discrete and continuous systems in a linear setting is well established. The non-linear random vibration problem still poses a challenge, since there is no universal analytical method available to solve them exactly. Under these circumstances, the role of approximate, analytical or numerical methods becomes central. The method of stochastic linearization is a widely popular technique amongst investigators. The reason for such a popularity likely lies in the fact that the method was suggested, apparently independently, by Booton [1] in 1953, in the United States; and by Kazakov [2], in 1954, in the former Soviet Union, and had a chance of widespread use both in the West and in the East. By our estimate, by now there are over 400 papers devoted to the stochastic linearization technique, with attendant numerous reviews (for the most recent ones the reader may consult the papers by Socha and Soong [3], Elishakoff and Zhang [4], Elishakoff [5] and Prandlwarter and Schuëller [6]), and apparently a single monograph, by Roberts and Spanos [7], whose central theme is the stochastic linearization technique.

This study revisits the pioneering contribution by Booton [1] and demonstrates that the paper contains a subtle error. This error is corrected and, hopefully, error-free results are reported.

2. BOOTON'S LINEARIZATION TECHNIQUE

In order to maintain maximum closeness to the analysis by Booton [1] his notation will be adopted. One first considers a memoryless non-linear system. In such a system an output x_{AR} depends upon the instantaneous value of the input x_{AI} :

$$x_{AR} = f(x_{AI}). \tag{1}$$

One is interested in finding probabilistic characteristics of the response x_{AR} provided that those of the input x_{AI} are known. Booton suggested linearizing the relationship (1), by replacing it by a linear expression $K_{eq}\, x_{AI}$. In order to determine the value of the coefficient K_{eq}, Booton demanded that the mean-square difference

$$M = E[X_{AR} - K_{eq}\, X_{AI}]^2 \tag{2}$$

† Dedicated to the memory of Professor Dr. Ir Warner Tjrdus Koifer.

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must attain a minimal value. In equation (2), $E[\ ]$ means mathematical expectation. The value of K_{eq} that minimizes equation (2) was referred to by Booton as *equivalent gain*. Since

$$M = E[X_{AR}^2] - 2K_{eq}\, E[X_{AI}\, X_{AR}] + K_{eq}^2\, E[X_{AI}^2], \tag{3}$$

the minimization requirement

$$\mathrm{d}M/\mathrm{d}K_{eq} = 0, \tag{4}$$

taking into account that $x_{AR} = f(x_{AI})$, results in

$$K_{eq} = E[X_{AI}\, X_{AR}]/E[X_{AI}^2] = E[X_{AI} f(X_{AI})]/E[X_{AI}^2]. \tag{5}$$

With $p(x)$ denoting the probability density of the input,

$$K_{eq} = \int_{-\infty}^{\infty} xf(x)p(x)\,\mathrm{d}x \Bigg/ \int_{-\infty}^{\infty} x^2 p(x)\,\mathrm{d}x. \tag{6}$$

Booton [1] considered a specific case of a non-linear "sharp" limiter. If f_{max} denotes the limiting level and x_{LI} denotes the input, the non-linear function f was specified as

$$f(x_{LI}) = \begin{cases} -f_{max} & \text{for} \quad x_{LI} < -f_{max} \\ x_{LI} & \text{for} \quad -f_{max} < x_{LI} < f_{max} \\ f_{max} & \text{for} \quad f_{max} < x_{LI} \end{cases}. \tag{7}$$

For the Gaussian probability density for x_{LI},

$$p(x_{LI}) = \frac{1}{\sigma_{LI}\sqrt{2\pi}} \exp\left[-\frac{x_{LI}^2}{2\sigma_{LI}^2}\right], \tag{8}$$

with σ_{LI} denoting the standard deviation, the equivalent gain is obtained as

$$K_{eq} = E[x_{LI} f(x_{LI})/E[x_{LI}^2] = \Psi\left(\frac{f_{max}}{\sigma_{LI}\sqrt{2}}\right) \tag{9}$$

where $\Psi(z)$ is an error function

$$\Psi(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t. \tag{10}$$

Equations (1)–(10) are recapitulated from Booton's paper for the convenience of the reader. Booton made a subtle error, however, when he extended the stochastic linearization analysis to a system with memory.

3. BOOTON'S PROBLEM

A simple non-linear dynamic system (feedback control system) is considered, governed by the equation

$$\mathrm{d}x/\mathrm{d}t + kf(x) = k\,\mathrm{d}z/\mathrm{d}t. \tag{11}$$

In equation (11), k is a positive coefficient, and $z(t)$ is a stationary Gaussian random excitation with spectral density, for an assigned value of the constant Φ_0, given by

$$\Phi_z = \Phi_0 \,|(\mathrm{i}\omega)^2/\omega_I^2 + 2\zeta_I \,\mathrm{i}\omega/\omega_I + 1|^{-2}. \tag{12}$$

Booton utilized the same non-linear function $f(x)$ as given by equation (7). He replaced the non-linear system by a linear one

$$\mathrm{d}x/\mathrm{d}t + kK_{eq}\, x = k\; \mathrm{d}z/\mathrm{d}t, \tag{13}$$

where the value of equivalent gain K_{eq} in equation (13) was postulated as one given by equation (9). However, when deriving equation (9), the input density with its parameter σ_{LI} was known. Here, the probability density of x, which serves as an *input* to the non-linear transformation defined as $y = f(x)$, is unknown at this stage. Rather, the standard deviation σ_x of the effective system is evaluated on the basis of the response of the *entire* linear system. Let K_{eq} therefore be left as an unknown in equation (13) and evaluate the mean square value of the response x:

$$\sigma_x^2 = \int_{-\infty}^{\infty} \Phi_x(\omega)\, \mathrm{d}\omega = \int_{-\infty}^{\infty} \left|\frac{k\mathrm{i}\omega}{\mathrm{i}\omega + kK_{eq}}\right|^2 \Phi_z(\omega)\, \mathrm{d}\omega, \tag{14}$$

where $\Phi_x(\omega)$ is the spectral density of the response x. Now one wants to find K_{eq} from the criterion postulated by Booton, i.e., through minimization of mean-square difference between the non-linear restoring force and its linear counterpart:

$$E[(f(x) - K_{eq}\, x)^2] = \min. \tag{15}$$

This is achieved by requiring that the derivative of the left side of equation (15) vanishes:

$$(\mathrm{d}/\mathrm{d}K_{eq}) E[(f(x) - K_{eq}\, x)^2] = 0 \tag{16}$$

or

$$(\mathrm{d}/\mathrm{d}K_{eq})\{E[f^2(x)] - 2K_{eq}\, E[xf(x)] + K_{eq}^2\, E[x^2]\} = 0. \tag{17}$$

Since in the new circumstances the probabilistic characteristics $E[f^2(x)]$, $E[xf(x)]$, and $E[x^2]$ *depend* on K_{eq}, the result of the differentiation should reflect this fact. Therefore, equation (17) is equivalent to

$$\frac{\mathrm{d}E[f^2(x)]}{\mathrm{d}K_{eq}} - 2K_{eq}\frac{\mathrm{d}E[xf(x)]}{\mathrm{d}K_{eq}} + K_{eq}^2 \frac{\mathrm{d}E[x^2]}{\mathrm{d}K_{eq}} - 2E[xf(x)] + 2K_{eq}\, E[x^2] = 0 \tag{18}$$

As one has seen, σ_x^2 depends upon K_{eq}. To stress this dependence, one denotes the mean-square value as $\sigma_x^2(K_{eq})$. Analogously, the probabilistic characteristics of $f^2(x)$, $xf(x)$ depend upon K_{eq}. Therefore, in the correct setting, all terms in equation (18) are non-vanishing.

4. BOOTON'S ERROR AND MODIFIED DERIVATIONS

If the first three terms are neglected, the Booton's equation for the equivalent gain, namely, equation (5), that he also used for the system with memory is obtained. For the dynamic system with memory governed by equation (11), the standard deviation of the response, appearing in equation (9), has to be evaluated on the basis of equation (14). It yields the following expression for the mean square value of the response:

$$E[x^2] = \sigma_x^2(K_{eq}) = \frac{k^2 I^2}{[1 + 2\zeta_I\, kK_{eq}/\omega_I + (kK_{eq}/\omega_I)^2]}, \tag{19}$$

where I is the intensity of the input of the system, and is equal to

$$I^2 = \int_{-\infty}^{\infty} \Phi_z(\omega)\, \mathrm{d}\omega. \tag{20}$$

Therefore, Booton's results are obtained by numerical solution of equation (9) in conjunction with equation (19).

Generally, however, the derivatives in the first three terms in equation (18) do not vanish and therefore, are not negligible. The derivative of mean square value of the response (equation 19) reads

$$\mathrm{d}E[x^2]/\mathrm{d}K_{eq} = -2(\sigma_x^4/k^2I^2)\left(\frac{\zeta_I k}{\omega_I} + \frac{k^2K_{eq}}{\omega_I^2}\right). \tag{21}$$

The mean square value of the force provided by the ''sharp'' limiter defined by equation (7), taking into account equations (8)–(10) is

$$\begin{aligned} E[f^2(x)] &= 2\left[\int_0^{f_{max}} x^2 p(x)\,\mathrm{d}x + f_{max}^2 \int_{f_{max}}^{\infty} p(x)\,\mathrm{d}x\right] \\ &= \sigma_x^2\,\Psi\left(\frac{f_{max}}{\sigma_x\sqrt{2}}\right) - \frac{\sqrt{2}}{\sqrt{\pi}}\sigma_x f_{max}\exp\left[-\frac{f_{max}^2}{2\sigma_x^2}\right] + f_{max}^2\left[1-\Psi\left(\frac{f_{max}}{\sigma_x\sqrt{2}}\right)\right]. \end{aligned} \tag{22}$$

Therefore, one gets

$$\frac{\mathrm{d}E[f^2(x)]}{\mathrm{d}K_{eq}} = \frac{2\sigma_x^4}{k^2I^2}\left(\frac{\zeta_I k}{\omega_I} + \frac{k^2K_{eq}}{\omega_I^2}\right)\left\{\sqrt{\frac{2}{\pi}}\frac{f_{max}}{\sigma_x}\left(1-\frac{f_{max}^2}{\sigma_x^2}\right)\exp\left(-\frac{f_{max}^2}{2\sigma_x^2}\right) - 1\right\}. \tag{23}$$

Moreover, in perfect analogy with equation (9),

$$E[xf(x)] = \sigma_x^2\,\Psi\left(\frac{f_{max}}{\sigma_x\sqrt{2}}\right). \tag{24}$$

The derivative of equation (24), appearing in equation (16), reads

$$\frac{\mathrm{d}E[xf(x)]}{\mathrm{d}K_{eq}} = \frac{\sigma_x^4}{k^2I^2}\left(\frac{\zeta_I k}{\omega_I} + \frac{k^2K_{eq}}{\omega_I^2}\right)\left\{-2\Psi\left(\frac{f_{max}}{\sigma_x\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}}\frac{f_{max}}{\sigma_x}\exp\left[-\frac{f_{max}^2}{2\sigma_x^2}\right]\right\}. \tag{25}$$

Finally, the equation for the correct value of the equivalent *gain* K_{eq} is obtained by substituting of equations (21)–(25) into equation (17):

$$\begin{aligned} &\left(\frac{\zeta_I k}{\omega_I} + \frac{k^2K_{eq}}{\omega_I^2}\right)\left\{\frac{\sigma_x}{kI}\left\langle\sqrt{\frac{2}{\pi}}\frac{f_{max}}{kI}\exp\left[-\frac{f_{max}^2}{2\sigma_x^2}\right]\left(1-\frac{f_{max}^2}{2\sigma_x^2}\right) - \frac{\sigma_x}{kI}\right\rangle - 2\sqrt{2}K_{eq}\frac{\sigma_x}{kI}\left[\sqrt{2}\frac{\sigma_x}{kI}\Psi\right.\right. \\ &\quad\left.\left.\times\left(\frac{f_{max}}{\sigma_x\sqrt{\pi}}\right) + \frac{1}{\sqrt{\pi}}\frac{f_{max}}{kI}\exp\left[-\frac{f_{max}^2}{kI}\right] - \frac{2K_{eq}^2\sigma_x^2}{k^2I^2}\right]\right\} - \Psi\left(\frac{f_{max}^2}{\sigma_x}\right) + 2K_{eq} = 0. \end{aligned} \tag{26}$$

The results descending from numerical solution of equation (9) (Booton's solution) and equation (26) (the one suggested by the authors) are discussed in the next section.

5. NUMERICAL RESULTS

In Figure 1 the value of the mean square difference between the force of the sharp limiter and its equivalent counterpart, normalized with respect to $(I\omega_I)^2$, versus the gain of the equivalent linear system K_{eq} is depicted for $\zeta = 0{\cdot}7$, the normalized limiter level $f_{max}/(\omega_{II}) = 0{\cdot}6$ and k/ω_I fixed at the unity. The results stemming from Booton solution (equation (9)) and from equation (26) are indicated by stars. It can be recognized that the Booton technique does not lead to a minimization of the mean square difference, while equation (26) allows this aim to be attained, as expected.

In Figures 2(a–c) the curves of the normalized mean square force difference versus the normalized coefficient k/ω_I are compared for equivalent gain obtained via the Booton solution (solid line) or present equation (26) (broken line). The values of the normalized limiter level $f_{max}/(\omega_I I)$ are assumed equal to 0·2 or 0·4 (Figure 2(a)), equal 0·6 (Figure 2(b)) or equal to 0·8 or 1 (Figure 2(c)), respectively. Figure 2(a) shows that for small values of $f_{max}/(\omega_I I)$ (0·2 and 0·4) the two procedures result in almost the same value of the mean square force difference; the curves relating to the Booton solution and the one stemming from equation (26) are superimposed. When $f_{max}/(\omega_I I) = 0{\cdot}6$ (Figure 2(b)), a slight difference between the two curves can be recognized. As expected, equation (26) leads to values of mean square force difference that are smaller than the counterpart obtained by Booton, for all the values of the coefficient k. This behavior is confirmed by the curves obtained for $f_{max}/(\omega_I I) = 0{\cdot}8$ and 1 (Figure 2(c)). The greater is the influence of the sharp limiter on the response x, i.e., the greater $f_{max}/(\omega_I I)$ and k/ω_I are, the greater is the gap between the mean square force difference stemming from Booton's solution and the equation (26) solution.

Figures 3(a–c) depict the normalized standard deviation value $\sigma_x/(kI)$ of the response of Booton's dynamic system defined as [2]

$$\sigma_x/(kI) = [1 + 2\zeta_I K_{eq} k/\omega_I + (kK_{eq}/\omega_I)^2]^{-1/2} \tag{27}$$

versus the normalized coefficient k/ω_I with the normalized limiter level $f_{max}/(\omega_I I)$ as a parameter. The results of stochastic linearization technique, developed according to Booton's formulation (broken line ––––) and by equation (26) (dotted line · · · · ·) are compared with the responses of the original non-linear system (solid line) obtained by the

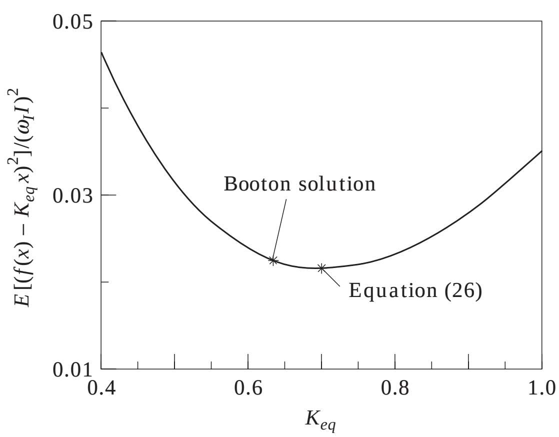

Figure 1. Variation of normalized mean-square force difference versus the gain of the equivalent linear system; $k/\omega_I = 1$; $f_{max}/(\omega_I I) = 0{\cdot}6$.

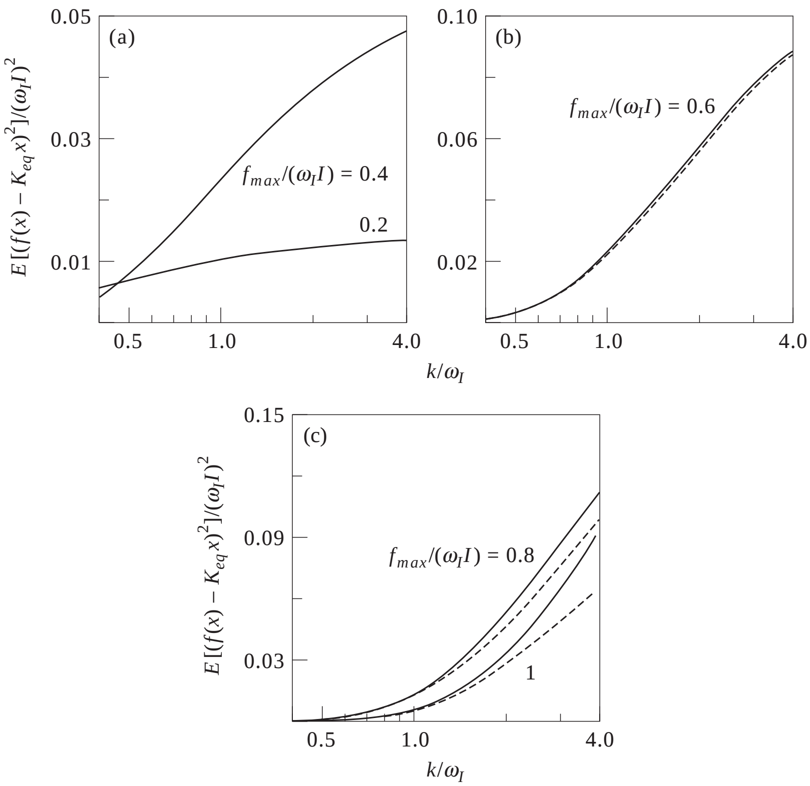

Figure 2. Normalized mean-square force difference for the equivalent gain given by Booton's linearization (——); proposed linearization (· · ·). (a) $f_{max}/(\omega_I I) = 0{\cdot}2$ and 0·4; (b) 0·6, (c) 0·8 and 1.

Monte Carlo simulation. The latter is performed by considering 1000 samples of the input process characterized by the power spectral density given in equation (12). The generation of the samples is conducted by harmonic wave superposition, namely, by the procedure proposed by Shinozuka and Jan in reference [8]. By utilization of the ergodicity property of the response, 500 values of the stationary phase of the response of each sample are considered. Therefore, each point of the curves in the figure are obtaining by averaging of 500 000 values. The step-by-step integration of the non-linear equation (equation (11)) is carried out by the Newmark numerical integration method [9].

In Figure 3(a) values of the normalized limiter level $f_{max}/(\omega_I I)$ equal to 0·2 or 0·4 are considered. The curves show that stochastic linearization leads to an underestimation of the response of the original non-linear system. Moreover, the correct minimization of the mean square force difference leads to the values of the response parameter that are below the ones given by the Booton technique, and therefore *farther* from the exact solution, even if the mean square force difference is smaller.

Such an interesting behaviour is confirmed by the curves portrayed in Figure 3(b) and Figure 3(c), where the value of the normalized limit level, 0·6, 0·8, or 1, are considered, respectively. However, when the normalized limit level approaches infinity (Figure 3(c)), the behavior of the sharp limiter tends to that of the linear elastic system. Therefore, both Booton's procedure and equation (26), leading to $k = 1$, allow the response of the original (linear) system to be obtained.

Remark: A natural question arises: does Booton's erroneous linearization procedure yield results that are *closer* to correct solution than those obtained by correct linearization for *all* mechanical systems? The desire to answer this question led the authors to consider another simple dynamic system, namely, the following half-degree-of-freedom system, governed by the equation [4, 10]

$$\mathrm{d}x/\mathrm{d}t + \varepsilon\,\mathrm{sign}\,(x) = Z(t), \tag{28}$$

where ε is a positive constant and $Z(t)$ is a Gaussian white noise with zero mean and intensity I_0. The exact solution for the probability density function of the response, given by Caughey and Dienes [11], and by Bolotin [10] reads

$$p_x\,(x) = \frac{\varepsilon}{I_0}\exp\left(-\frac{2\varepsilon|x|}{I_0}\right). \tag{29}$$

Therefore, the exact value of mean square response is

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_x\,(x)\,\mathrm{d}x = \frac{I_0^2}{2\varepsilon^2}. \tag{30}$$

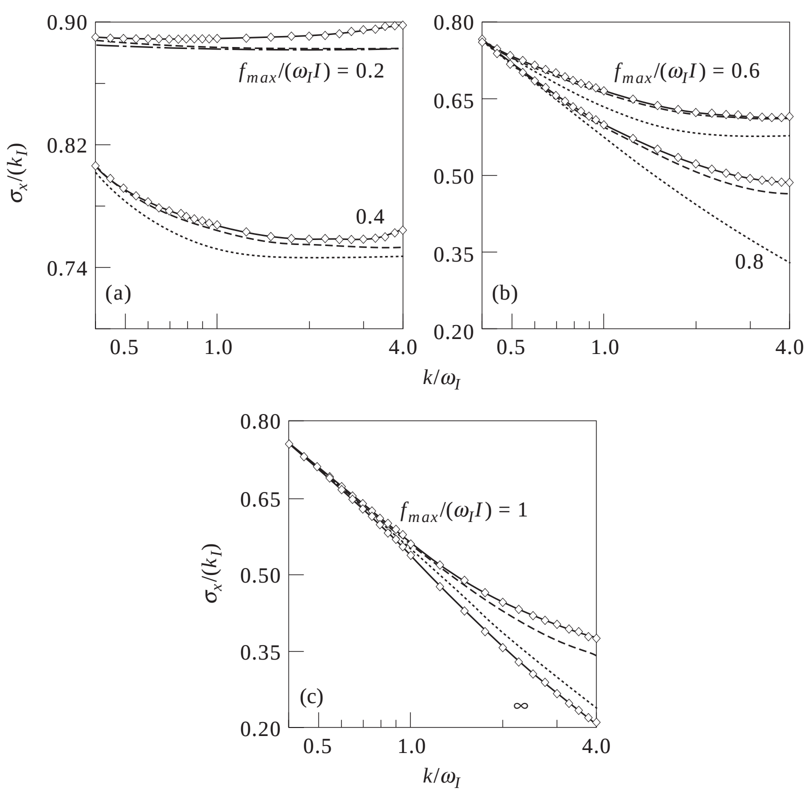

Figure 3. Comparison of normalized standard deviation of the response for original non-linear and linearized systems: – – – – Booton; · · · · equation (26); —○— Monte Carlo. (a) $f_{max}\,/(\omega_I\,I)$ = 0·2 and 0·4, (b) 0·6 and 0·8, (c) 1 and ∞.

According to linearization technique, the non-linear system is replaced by the equivalent linear one:

$$\mathrm{d}x/\mathrm{d}t + K_{eq}\, x = Z(t). \tag{31}$$

The coefficient K_{eq} is evaluated requiring that

$$(\mathrm{d}/\mathrm{d}K_{eq})\; E[(\varepsilon \text{ sign } (x) - K_{eq}\, x)^2] = 0. \tag{32}$$

According to the procedure proposed by Booton (see equation (6)), i.e., neglecting the dependence of statistical moment of the response upon K_{eq}, equation (32) is reduced to

$$K_{eq} = \varepsilon E[|x|]/E[x^2]. \tag{33}$$

The mean square value of the response, for the system governed by equation (31) reads

$$E[x^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{I_0}{|\mathrm{i}\omega + K_{eq}\,|^2}\,\mathrm{d}\omega = \frac{I_0}{2K_{eq}}. \tag{34}$$

Taking into account that the input is Gaussian and the replacing system is linear, the response is Gaussian, too. Therefore the numerator of equation (33) becomes

$$\varepsilon E[|x|] = \frac{\varepsilon}{\sqrt{2\pi E[x^2]}}\int_{-\infty}^{\infty} |x| \exp\left(\frac{-x^2}{2E[x^2]}\right)\mathrm{d}x = \varepsilon\sqrt{\frac{I_0}{K_{eq}\,\pi}}. \tag{35}$$

By substitution of equations (34) and (35) in equation (33), one gets

$$K_{eq} = 4\varepsilon^2/(\pi I_0) \tag{36}$$

and the attendant value of mean square displacement is

$$E[x^2] = \pi I_0^2\,/(8\varepsilon^2). \tag{37}$$

The procedure suggested in this study requires that equations (34) and (35) are substituted in equation (32), before that the derivatives are evaluated. Thus, one gets

$$(\mathrm{d}/\mathrm{d}K_{eq})\,[\varepsilon^2 + I_0\, K_{eq}\,/2 - 2\varepsilon\sqrt{I_0\, K_{eq}\,/\pi}] = 0, \tag{38}$$

which leads to the same value of K_{eq} obtained in equation (36). This implies that sometimes the erroneous realization of the linearization technique may result in the correct answer.

6. CONCLUSION

The application of the stochastic linearization technique to the specific problem analyzed by Booton is re-examined. It is shown that Booton has made a subtle error in the procedure for minimization of the mean square force difference between the sharp limiter and its linear equivalent counterpart. When he extended the procedure derived for a simple *memoryless* sharp limiter to a dynamic control system *with memory* he erroneously neglected the dependence of the statistical moments of the response upon the variable with respect to which minimization was conducted, namely, the gain of the equivalent linear system.

A new procedure, correcting the above error, is elucidated. It allows one to obtain the true minimal value of the mean square force difference. It must be stressed that the new procedure leads to results that are *farther* from the response of the original non-linear system than those obtained by Booton. This suggests that the stochastic linearization technique, based on minimization of mean square force difference of the non-linear system and its linear counterpart, has less accuracy than it has been previously stated in the literature.

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